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Ermakov invariant and the general solution for a damped harmonic oscillator with a force quadratic in velocity

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Abstract
This paper makes a generalization of the Ermakov system to obtain the Ermakov invariant for the case of a variable mass and variable frequency oscillator with a force quadratic in velocity, and hence find the general solution of a time dependent damped harmonic oscillator with a force quadratic in velocity.

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1. Introduction

During the past decades, several techniques to treat time dependent systems (TDS) were proposed: among these, the quantum invariant operator method, the propagator method and the separation of variables treatment are particularly well known. Recently we have discussed the invariants and propagator for a complex time dependent system with a force quadratic in velocity [1–4]. In the course of research on TDS, we find that there is an elegant method to obtain the invariant of TDS, the Ermakov method. Now we shall introduce this method [5] and make a generalization to obtain the so-called Ermakov invariant for the case of a variable mass and variable frequency oscillator with a force quadratic in velocity, and hence find the general solution of a time dependent damped harmonic oscillator with a force quadratic in velocity.

It is only now becoming known that the problem of the time dependent oscillator was first solved by Ermakov [6] in 1880. Ermakov obtained a first integral of the equation of motion for the harmonic oscillator with variable frequency

$$\ddot{q} + \omega^2(t)q = 0 \tag{1.1}$$

by introducing the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}. \tag{1.2}$$

Eliminating the ω^2 terms from equations (1.1) and (1.2), multiplying by the integrating factor $\rho\dot{q} - \dot{\rho}q$ and integrating the resulting differential equation, Ermakov obtained the invariant (first integral)

$$I = \frac{1}{2}\left[(\rho\dot{q} - \dot{\rho}q)^2 + \left(\frac{q}{\rho}\right)^2\right] \tag{1.3}$$

which is usually called the Lewis or Ermakov–Lewis invariant, after Lewis who rediscovered it in 1966 [7, 8]. A pair of ordinary differential equations which are linked by an invariant, such as equations (1.1) and (1.2), is now termed an Ermakov system and the invariant, such as equation (1.3), is termed an Ermakov invariant, which links two functions $x(t)$ and $\rho(t)$ and provides a nonlinear superposition law [9].

Beginning in 1979, Ray and Reid [5,9,10] generalized the pair of equations (1.1) and (1.2) to the following pair of equations:

$$\ddot{q} + \omega^2(t)q = g(\rho/q)/(\rho q^2) \tag{1.4}$$

and

$$\ddot{\rho} + \omega^2(t)\rho = f(q/\rho)/(q\rho^2). \tag{1.5}$$

By eliminating $\omega^2(t)$ from equations (1.4) and (1.5) they obtain the invariant

$$I = \tfrac{1}{2}[\phi(q/\rho) + \theta(\rho/q) + (\rho\dot{q} - \dot{\rho}q)^2] \tag{1.6}$$

where $g(\rho/q)$ and $f(q/\rho)$ are arbitrary functions of their arguments, and $\phi(q/\rho) = 2\int^{q/\rho} f(v)\,\mathrm{d}v$, $\theta(\rho/q) = 2\int^{\rho/q} g(w)\,\mathrm{d}w$. Now we will show that the Ermakov system can be further generalized. Since there has been considerable interest in a particle with a force quadratic in the velocity, we shall consider the general case of the problem of a time dependent mass and time dependent frequency oscillator with a force quadratic in velocity. In the following section we shall make a further generalization of the Ermakov method to treat this rather general TDS.

2. A generalization of the Ermakov system

For the case of a variable mass and variable frequency oscillator with a force quadratic in velocity, the equation of motion is [1]

$$\ddot{x} + \beta\dot{x} + \frac{1}{2}\gamma\dot{x}^2 + \frac{\partial V}{\partial x} = 0. \tag{2.1}$$

Now we introduce an arbitrary function $u(x)$ and write $\frac{\partial V}{\partial x} = \omega^2(t)u(\frac{\mathrm{d}u}{\mathrm{d}x})^{-1}$. The reason for introducing such an arbitrary function is the following consideration, that in the particular case $u = Cx(C = \text{const})$, $\frac{\partial V}{\partial x} = \omega^2(t)x$, $\frac{\partial V}{\partial x}$ corresponds to a harmonic potential with time dependent frequency. By the requirement of the existence of an invariant we must have [11]

$$\frac{1}{2}\gamma = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}. \tag{2.2}$$

In this case we find the Ermakov system:

$$\ddot{x} + \beta\dot{x} + \dot{x}^2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \omega^2(t)u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1} = 0 \tag{2.3}$$

and

$$\ddot{\rho} + \Omega^2(t)\rho = \Omega_0^2/\rho^3 \tag{2.4}$$

where $\beta = 2\epsilon = \frac{\dot{M}(t)}{M(t)}$, $M(t)$ is the time dependent mass, $\Omega^2(t) = \omega^2(t) - \epsilon^2 - \dot{\epsilon}$, $\omega(t)$ is the time dependent frequency and Ω_0 is a constant. Eliminating $\omega^2(t)$ from equations (2.3) and (2.4) we obtain

$$\ddot{x} + \beta\dot{x} + \dot{x}^2 \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1} \left(\epsilon^2 + \dot{\epsilon} + \frac{\Omega_0^2}{\rho^4} - \frac{\ddot{\rho}}{\rho}\right) = 0. \tag{2.5}$$

Multiplying equation (2.5) by the factor $\rho M^{1/2}(\mathrm{d}u/\mathrm{d}x)$ and rewriting this equation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho M^{1/2}\dot{x}(\mathrm{d}u/\mathrm{d}x) - \dot{\rho} M^{1/2} u + \epsilon\rho M^{1/2} u] + \frac{\Omega_0^2 u M^{1/2}}{\rho^3} = 0. \tag{2.6}$$

Multiplying equation (2.6) by the factor $[\rho M^{1/2}\dot{x}(\mathrm{d}u/\mathrm{d}x) - (\dot{\rho} - \epsilon\rho)M^{1/2}u]$ and integrating it, we readily obtain the Ermakov invariant for this case:

$$I = \frac{1}{2}\left\{\left(\frac{\Omega_0 u M^{1/2}}{\rho}\right)^2 + [\rho M^{1/2}\dot{x}(\mathrm{d}u/\mathrm{d}x) - (\dot{\rho} - \epsilon\rho)M^{1/2}u]^2\right\}. \tag{2.7}$$

When $u = Cx (C = \text{const})$, $M(t) = M = \text{const}$, $\Omega_0 = 1$ equations (2.3), (2.4) and (2.7) reduce to equations (1.1)–(1.3). From equation (2.7) we obtain

$$I = \frac{1}{2}\left\{\left(\frac{\Omega_0 u M^{1/2}}{\rho}\right)^2 + \left[\frac{\mathrm{d}(\frac{\Omega_0 u M^{1/2}}{\rho})}{\frac{\Omega_0\,\mathrm{d}t}{\rho^2}}\right]^2\right\}. \tag{2.8}$$

Let $z = \frac{\Omega_0 u M^{1/2}}{\rho}$ and $\mathrm{d}\phi = \frac{\Omega_0\,\mathrm{d}t}{\rho^2}$. Then equation (2.8) becomes

$$I = \frac{1}{2}\left\{z^2 + \left[\frac{\mathrm{d}z}{\mathrm{d}\phi}\right]^2\right\}. \tag{2.9}$$

Solving equation (2.9) we get

$$z = \sqrt{2I}\sin(\phi + c) \tag{2.10}$$

where c is a constant. From equation (2.10) we get

$$u(x) = \rho M^{-\frac{1}{2}}(A\cos\phi + B\sin\phi) \tag{2.11}$$

where ρ is a particular solution of the auxiliary equation (2.4), and A and B are constants. From equation (2.11) we readily obtain the general solution of equation (2.3).

3. Time dependent damped harmonic oscillator with a force quadratic in velocity

If we put

$$u(x) = \frac{2}{\gamma}\left[\exp\left(\frac{\gamma x}{2}\right) - 1\right] \tag{3.1}$$

then

$$\frac{\partial V}{\partial x} = \omega^2(t)\frac{1 - \exp[-(\gamma/2)x]}{\gamma/2}. \tag{3.2}$$

This is just the case we discussed in [1]. For this case, when $\gamma \to 0$, $\frac{\partial V}{\partial x} \to \omega^2 x$, then equation (2.1) is just the equation of motion for a damped harmonic oscillator with a force quadratic in velocity. Substituting equation (3.1) into equation (2.11) we readily obtain the general solution for this case:

$$x(t) = \frac{2}{\gamma}\ln\left[\frac{\gamma}{2}\rho M^{-\frac{1}{2}}(A\cos\phi + B\sin\phi) + 1\right]. \tag{3.3}$$

When $\gamma \to 0$, we can neglect terms of order higher than γ^2, then equation (3.3) becomes

$$x(t) = \rho M^{-\frac{1}{2}}(A\cos\phi + B\sin\phi) - \frac{\gamma\rho^2}{4M}(A\cos\phi + B\sin\phi)^2 \tag{3.4}$$

where ρ is any particular solution of equation (2.4).

4. A further generalization of the Ermakov system

From the inspiration of the generalization from equations (1.1) and (1.2) to equations (1.4) and (1.5), we can further generalize equations (2.3) and (2.4) to

$$\ddot{x} + \beta\dot{x} + \dot{x}^2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \omega^2(t)u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1} = \frac{g(\frac{\rho}{M^{1/2}u})}{\rho M^{1/2}\frac{\mathrm{d}u}{\mathrm{d}x}(M^{1/2}u)^2} \tag{4.1}$$

and

$$\ddot{\rho} + \Omega^2(t)\rho = \frac{1}{M^{1/2}u\rho^2}f\left(\frac{M^{1/2}u}{\rho}\right). \tag{4.2}$$

The corresponding Ermakov invariant can be found to be

$$I = \tfrac{1}{2}[\rho M^{1/2}\dot{x}(\mathrm{d}u/\mathrm{d}x) - (\dot{\rho} - \epsilon\rho)M^{1/2}u]^2 + \int^{\frac{M^{1/2}u}{\rho}} f(v)\,\mathrm{d}v + \int^{\frac{\rho}{M^{1/2}u}} g(w)\,\mathrm{d}w. \tag{4.3}$$

Since $u(x)$ is an arbitrary function of x, $f(v)$ and $g(w)$ are arbitrary functions of their argument. Hence this generalized Ermakov system should have many applications both in finding invariants and in solving nonlinear differential equations by using the nonlinear superposition law. Obviously, when $u = Cx(C = \text{const})$, $M(t) = \text{const}$, equations (4.1)–(4.3) reduce to equations (1.4)–(1.6).

5. Conclusion

The question of the existence of invariants is one of central importance in the study of any dynamical system, especially for the TDS. Since the quantum invariant operator method [12] is particularly useful in treating time dependent systems in quantum mechanics, the Ermakov method and its generalizations will be of great value in finding invariants of some interesting TDS. In this paper we have discussed the case of a variable mass and variable frequency oscillator with a force quadratic in velocity. This force term has been considered by some authors [13] and it is well known that the case of a variable mass and variable frequency has been discussed by quite a few authors [14,15]. Thus we have shown that Ermakov's method is very powerful for finding the invariant of this very complex TDS. The further generalizations introduced in the previous section will give further applications in many interesting TDS.

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